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Compact three-dimensional QED: A simple example of a variational calculation in a gauge theory

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We apply a simple mean-field-like variational calculation to compact QED in $2+1$ dimensions. Our variational ansatz explicitly preserves the compact gauge invariance of the theory. We reproduce in this framework all the known results, including dynamical mass generation, Polyakov scaling, and the nonzero string tension. It is hoped that this simple example can be a useful reference point for applying similar approximation techniques to non-Abelian gauge theories.

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I. INTRODUCTION

Study of the confining regime in QCD, as well as of many other strong interaction phenomena, requires going beyond the perturbation theory. However, the application of analytic nonperturbative methods in quantum field theory is a very complicated and not too well-developed area. This is especially true for non-Abelian gauge theories. Recently we have formulated a gauge-invariant variational approach and used a restricted variational ansatz to study the ground state of a pure Yang Mills theory in $3+1$ dimensions [1]. As with any new technique, it is desirable to develop an intuition for it, by first considering simpler examples.

A theory which possesses many common qualitative features with QCD (such as confinement and dynamical mass generation), and yet is much simpler and much more tractable is compact electrodynamics in $2+1$ dimensions. Moreover, this theory has been previously extensively studied by both analytical [2] and numerical [3] methods. It seems therefore to be a perfect test ground for the application of our variational method. This is precisely the aim of this note. We will apply the gauge-invariant variational approximation of [1] to this theory. It is hoped that this toy calculation can teach us something about improving the variational ansatz for realistic $(3+1)$-dimensional non-Abelian theories. It is also a nice exercise in itself, since it gives a vivid Hamiltonian picture of Polyakov's monopole-instanton condensation phenomenon, which to our knowledge does not exist in the literature.

The paper is organized as follows. In the rest of this section we discuss the Hamiltonian formalism for three-dimensional compact QED(QED$_3$). In Sec. II, we set up our variational ansatz and discuss some important properties of the variational wave functionals. Section III contains calculation of the expectation value of the energy and solution of minimization equations. We also show that the Wilson loop in the best variational state has the area law and calculate the string tension. In Sec. IV, we discuss our results and the interpretation of our calculation from the point of view of Polyakov's dilute monopole gas approximation to Euclidean partition function.

The theory is defined by the Hamiltonian

$$H = \tfrac{1}{2}[E_i^2 + b^2] \ . \tag{1.1}$$

The field b is somewhat different from the usual magnetic field $B = \epsilon_{ij}\partial_i A_j$. We will explain its definition in a short while. All the physical states should satisfy the Gauss law constraint

$$\exp\left(i\int d^2x\, \partial_i\lambda(x)E_i(x)\right)|\Psi\rangle = |\Psi\rangle \ . \tag{1.2}$$

One should note that there is a crucial difference between the Gauss law in the compact theory and in the noncompact one. In the noncompact theory equation (1.2) should be satisfied only for regular functions λ. For example, the operator

$$V(x) = \exp\left\{\frac{i}{g}\int d^2y \frac{\epsilon_{ij}(x-y)_j}{(x-y)^2}E_i(y)\right\} , \tag{1.3}$$

which has the form of (1.2) with the function λ proportional to the planar angle θ, $\lambda = (1/g)\theta(x)$, does not act trivially on physical states. In fact, this operator creates pointlike magnetic vortices with magnetic flux $2\pi/g$ [4] and therefore changes the physical state on which it is acting.

In the compact theory the situation in this respect is quite different. Pointlike vortices with a quantized magnetic flux $2\pi n/g$ cannot be detected by any measurement. In Euclidean path-integral formalism of [2] this

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is the statement that the Dirac string of the monopole is unobservable and does not cost any (Euclidean) energy. In the Hamiltonian formalism this translates into the requirement that the creation operator of a pointlike vortex must be indistinguishable from the unit operator. In other words, the operator (1.3) generates a transformation which belongs to the compact gauge group, and should therefore act trivially on all physical states. Equation (1.2) should therefore be satisfied also for these operators.

Accordingly, the Hamiltonian of the compact theory also must be invariant under these transformations. The magnetic field defined as $B = \epsilon_{ij}\partial_i A_j$, on the other hand, does not commute with $V(x)$:

$$V^\dagger(x)B(y)V(x) = B(y) + \frac{2\pi}{g}\delta^2(x-y) \ . \qquad (1.4)$$

The Hamiltonian should therefore contain not B^2 but rather only its singlet part. This is the meaning of the field b in Eq. (1.1). Formally

$$b^2 = PB^2P \ , \qquad (1.5)$$

where P is the projection operator on the whole compact gauge group, which includes $V(x)$. This form is convenient for the purposes of the present calculation, and we therefore do not write down a more explicit expression for[1] b^2.

II. THE VARIATIONAL ANSATZ

Our aim in this paper is to find a vacuum wave functional of this theory. Following Polyakov, we will be working in the weakly coupled regime. Since the coupling constant g^2 in $2+1$ dimensions has dimension of mass, weak coupling means that the following dimensionless ratio is small:

$$\frac{g^2}{\Lambda} \ll 1 \ . \qquad (2.1)$$

Here Λ is the ultraviolet cutoff in the momentum space, which as always has to be introduced to regularize a quantum field theory.[2] For a weakly coupled theory one expects the vacuum wave functional (VWF) to be not too different from the vacuum of a free theory. Since the VWF of free (noncompact) electrodynamics is Gaussian in the field basis,

$$\psi[A_i] = \exp\left\{-\frac{1}{2}\int d^2x\, d^2y A_i(x)G^{-1}_{ij}(x-y)A_j(y)\right\} \ , \qquad (2.2)$$

Gaussian variational approach in this case should give a good approximation. An important caveat, however, is that the ground-state WF should be gauge invariant under the full compact gauge group. As a result it turns out that one cannot take just a Gaussian in A_i, since this will not preserve gauge invariance. The simplest generalization of the Gaussian ansatz which we use along the lines of [1], is to project a Gaussian WF into the gauge-invariant subspace of the Hilbert space. We therefore take as our variational ansatz the set of states

$$\Psi[A_i] = \int D\phi \exp\left\{-\frac{1}{2}\int d^2x\, d^2y \left[A_i(x) - \frac{1}{g}\partial_i\phi(x)\right] G^{-1}(x-y)\left[A_i(y) - \frac{1}{g}\partial_i\phi_i(y)\right]\right\} \ . \qquad (2.3)$$

The functional integral is over the phase function $\phi(x)$, and correspondingly the derivatives of ϕ in the exponential are understood modulo 2π. That is, these derivatives do not feel quantized discontinuities in $\phi(x)$. The mathematically more precise way to write this is to substitute $\partial_i\phi(x)$ by $\exp\{-i\phi(x)\}\partial_i\exp\{i\phi(x)\}$. We will use however, the above shorthand notation for convenience. The simple rotational structure of $G_{ij} = \delta_{ij}G$ that appears in the variational wave functional (2.3) is consistent with perturbation theory, as discussed in [1].

The ansatz (2.3) depends on one function $G(x)$. We now have to calculate the expectation value of the energy in this state, and then minimize it with respect to $G(x)$.

Before proceeding with the calculation we make the following (obvious) comment. The trial wave functional (2.3) has a simple interpretation from the point of view of states of a noncompact theory. To see this let us rewrite the functional measure $D\phi$ in a slightly different way. Any angular function $\phi(x)$ can be parametrized as

$$\phi(x) = \tilde{\phi}(x) + \phi_v(x) \ , \qquad (2.4)$$

where $\tilde{\phi}(x)$ is a smooth function and $\phi_v(x)$ contains all the discontinuities and can be written as

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[1]In the standard Euclidean lattice formulation of the theory the potential energy has the form $\cos(ga^2B)$, where a is the lattice spacing. This obviously has the same property as b^2, that is invariant under the transformation Eq. (1.4) and reduces to it in the weak-coupling limit. Our definition, Eq. (1.5), is equivalent to the Villian form of the action.

[2]If one considers compact U(1) as an unbroken sector of the spontaneously broken SU(2) (Georgi-Glashow model), the ultraviolet cutoff Λ will be proportional to the scale of the symmetry breaking. To be more precise, it is of the order of magnitude of the charged vector boson mass M_W.

$$\phi_v(x) = \sum_{\alpha=1}^{n_+} \theta(x - x_\alpha) - \sum_{\beta=1}^{n_-} \theta(x - x_\beta) \tag{2.5}$$

and $\theta(x - x_\alpha)$ is a polar angle on plane with a center at x_α. The functional measure can be written as

$$\int D\phi = \int D\tilde{\phi} \sum_{n_+=1}^{\infty} \sum_{n_-=1}^{\infty} \sum_{\{x_\alpha\}} \sum_{\{x_\beta\}} = \int D\tilde{\phi} \sum_{n_+=0}^{\infty} \sum_{n_-=0}^{\infty} \frac{1}{n_+! n_-!} \prod_{\alpha=1}^{n_+} \prod_{\beta=1}^{n_-} \left(\int d^2x_\alpha d^2x_\beta \Lambda^4 \right) , \tag{2.6}$$

where in the last equality we have just substituted integration over the coordinates of vortices and antivortices for summation, and for this reason introduced explicit UV cutoff Λ. We also introduced term with $n_+ = n_- = 0$ corresponding to the absence of vortices.

Let us define the function

$$\tilde{\Psi}[A] = \int D\tilde{\phi} \exp\left\{ -\frac{1}{2} \int d^2x\, d^2y \left[A_i(x) - \frac{1}{g}\partial_i\tilde{\phi}(x) \right] G^{-1}(x-y) \left[A_i(y) - \frac{1}{g}\partial_i\tilde{\phi}_i(y) \right] \right\} , \tag{2.7}$$

which differs from Ψ in that the integration is performed only over continuous gauge functions $\tilde{\phi}$. Obviously, $\tilde{\Psi}$ is invariant under noncompact gauge group, and therefore belongs to the Hilbert space of the noncompact theory. When acting on it, the vortex operator $V(x)$ defined in (1.3) just shifts $A_i(y)$ by $(1/g)\partial_i\theta(x-y)$. We have therefore the following representation for $\Psi[A]$:

$$\Psi[A] = \sum_{n_+, n_-=0}^{\infty} \prod_{\alpha=1}^{n_+} \prod_{\beta=1}^{n_-} V(x_\alpha) V^*(x_\beta) \tilde{\Psi}[A] . \tag{2.8}$$

This representation makes explicit the fact that a WF of the compact theory is constructed from a WF of the noncompact theory by taking a superposition of arbitrary number of vortices and antivortices at every point. This superposition is obviously invariant under multiplication by a vortex operator and therefore is its eigenfunction with eigenvalue 1. Having noted this, we now proceed to calculation of expectation values in the trial state (2.3) and the minimization of the vacuum expectation value of energy.

III. THE ENERGY MINIMIZATION

Let us for convenience introduce the notation

$$A_i^\phi(x) = A_i(x) - \frac{1}{g}\partial_i\phi(x) . \tag{3.1}$$

We will also switch to the matrix notations in the following, so that

$$A_i M A_i = \int d^2x\, d^2y\, A_i(x) M(x-y) A_i(y) . \tag{3.2}$$

The expectation value of any operator in the WF (2.3) is calculated as

$$\langle O \rangle = Z^{-1} \int D\phi' D\phi'' DA_i \exp\{-\tfrac{1}{2} A_i^{\phi'} G^{-1} A_i^{\phi'}\} O(A) \exp\{-\tfrac{1}{2} A_i^{\phi''} G^{-1} A_i^{\phi''}\} . \tag{3.3}$$

Here Z is the normalization factor, which is just the norm of the trial wave functional (2.3). Further, if the operator O is explicitly gauge invariant, we may shift the integration variable $A_i \to A_i^{\phi''}$, and reduce this expression to

$$\langle O \rangle = Z^{-1} \int D\phi\, D\eta\, DA_i \exp\{-\tfrac{1}{2} A_i^\phi G^{-1} A_i^\phi\} O(A) \exp\{-\tfrac{1}{2} A_i G^{-1} A_i\} , \tag{3.4}$$

where we have defined $\phi = \phi' - \phi''$ and $\eta = \phi' + \phi''$. The integral over η is trivial and just gives the volume of the gauge group. Since the same integral exactly enters Z, it always cancels between the numerator and denominator, and we shall omit it in the following.

As a first step let us calculate the normalization factor Z:

$$Z = \int D\phi\, DA_i \exp\{-\tfrac{1}{2}[A_i^\phi G^{-1} A_i^\phi + A_i G^{-1} A_i]\} . \tag{3.5}$$

The integration over A_i is Gaussian and can be trivially performed. The integral over the noncompact part of the gauge group $\tilde{\phi}$ is also Gaussian. Then

$$Z = Z_a Z_\phi Z_v \tag{3.6}$$

with

$$\begin{aligned} Z_a &= \det(\pi G)\,, \\ Z_\phi &= \int D\tilde{\phi}\exp\left\{-\frac{1}{4g^2}\int \partial_i\tilde{\phi}G^{-1}\partial_i\tilde{\phi}\right\} = \det\left[4\pi g^2\frac{1}{\partial^2}G\right]^{1/2}\,, \\ Z_v &= \int D\phi_v\exp\left\{-\frac{1}{4g^2}\int \partial_i\phi_v G^{-1}\partial_i\phi_v\right\}\,. \end{aligned} \tag{3.7}$$

Using Eqs. (2.5) and (2.7), one can represent Z_v as a partition function of the gas of vortices:

$$Z_v = \sum_{n_+,n_-=0}^{\infty}\prod_{\alpha=1}^{n_+}\prod_{\beta=1}^{n_-} dx_\alpha dx_\beta z^{n_++n_-}\exp\left\{-\frac{1}{4g^2}\left(\sum_{\alpha,\alpha'}D(x_\alpha - x_{\alpha'}) + \sum_{\beta,\beta'}D(x_\beta - x_{\beta'}) - \sum_{\alpha,\beta}D(x_\alpha - x_\beta)\right)\right\}\,. \tag{3.8}$$

The vortex-vortex interaction potential $D(x)$ and the vortex fugacity z are given by

$$\begin{aligned} D(x) &= 8\pi^2\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2}G^{-1}(k)\cos(kx)\,, \\ z &= \Lambda^2\exp\left(-\frac{1}{8g^2}D(0)\right)\,. \end{aligned} \tag{3.9}$$

Here $G(k)$ is the Fourier transform of the variational "propagator" $G(x)$.

Formula (3.8) reminds one of Polyakov's partition function of the monopole gas [2]. One should keep in mind, however, that the physical meaning of it is quite different. The gas described by Eq. (3.8) is two dimensional and not three dimensional, and the interaction between the particles is not Coulomb, but rather depends on the variational function G. Nevertheless, it has a definite relation with Polyakov's gas of monopoles and we will discuss this point in the last section of the paper.

Since $D(0)$ is singular, the last equation should be understood, as usual in the regularized sense, that is at finite UV cutoff, $D(0)$ should be substituted by $D(x = 1/\Lambda)$. Note that the variational function G explicitly appears in the vortex-vortex potential. Since we expect the UV behavior of $G(k)$ to be the same as in the free theory $[G^{-1}(k) \to k]$, we have

$$z = \Lambda^2\exp\left(-\frac{\pi}{2}\frac{\Lambda}{g^2}\right)\,. \tag{3.10}$$

In the following we will need to calculate correlation functions of the vortex density. To facilitate this we use the standard trick [2,5] to rewrite the partition function Z_v in terms of a path integral over a scalar field χ. Let us introduce the vortex density $\rho(x)$:

$$\rho(x) = \sum_{\alpha,\beta}[\delta(x - x_\alpha) - \delta(x - x_\beta)]\,. \tag{3.11}$$

The exponential factor in Eq. (3.9) (including the factors of fugacity) can then be rewritten as

$$\Lambda^{2(n_++n_-)}\int D\chi\exp\{-2g^2\chi D^{-1}\chi + i\rho\chi\}\,. \tag{3.12}$$

The summation over the number of vortices is trivial and gives

$$Z_v = \int D\chi\exp\left(-2g^2\chi D^{-1}\chi + \int_x 2\Lambda^2\cos\chi(x)\right)\,. \tag{3.13}$$

To calculate the correlator of ρ one can add $i\rho J$ to the vortex free energy, and calculate functional derivatives of the resulting partition function with respect to J at zero J. A simple derivation gives

$$\langle\rho(x)\rho(y)\rangle = 4g^2D^{-1}(x-y) - 16g^4\langle D^{-1}\chi(x)D^{-1}\chi(y)\rangle\,. \tag{3.14}$$

The propagator of χ is easily calculated. At weak coupling z is very small, and all our calculations will be performed to first order in z. To this order the only contribution comes from the tadpole diagrams. This is easily seen by rewriting the cosine potential in Eq. (3.13) in the normal ordered form[3]

$$\cos\chi = \langle\cos\chi\rangle_0 : \cos\chi := \frac{z}{\Lambda^2} : \cos\chi :\,. \tag{3.15}$$

Therefore, to first order in z,

$$\begin{aligned} \int d^2x\, e^{ikx}\langle\chi(x)\chi(0)\rangle &= \frac{1}{4g^2D^{-1}(k) + 2z} \\ &= \frac{D(k)}{4g^2} - z\frac{D^2(k)}{8g^4} + O(z^2)\,. \end{aligned} \tag{3.16}$$

[3]The normal ordering is performed relative to the free theory defined by the quadratic part of the action in Eq. (3.13).

The correlator of the vortex densities is then

$$K(k) = \int d^2x\, e^{ikx} \langle \rho(x)\rho(0) \rangle = 2z + O(z^2) \tag{3.17}$$

and in this approximation does not depend on momentum, the k dependence will appear in z^2 and higher-order terms.

Now we are ready to calculate the expectation value of the Hamiltonian (1.1). First, consider the electric part

$$\begin{aligned} \left\langle \int d^2x\, E_i^2 \right\rangle &= -\left\langle \int d^2x \frac{\delta^2}{\delta A_i^2} \right\rangle \\ &= Z^{-1} \int D\phi D A_i \exp\{-\tfrac{1}{2} A_i^{\phi} G^{-1} A_i^{\phi}\} [2\, \mathrm{tr} G^{-1} - A_i G^{-2} A_i] \exp\{-\tfrac{1}{2} A_i G^{-1} A_i\} \\ &= \mathrm{tr} G^{-1} - \frac{1}{4g^2} Z_{\phi}^{-1} Z_v^{-1} \int D\phi (\partial_i \phi G^{-2} \partial_i \phi) \exp\left\{ -\frac{1}{4g^2} \partial_i \phi G^{-1} \partial_i \phi \right\} . \end{aligned} \tag{3.18}$$

Performing the Gaussian integration over $\tilde{\phi}$, this reduces to

$$\begin{aligned} V^{-1} \left\langle \int E_i^2 \right\rangle &= \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} G^{-1}(k) - \frac{\pi^2}{g^2} \int \frac{d^2k}{(2\pi)^2} k^{-2} G^{-2}(k) K(k) \\ &= \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} G^{-1}(k) - \frac{2\pi^2}{g^2} z \int \frac{d^2k}{(2\pi)^2} k^{-2} G^{-2}(k) . \end{aligned} \tag{3.19}$$

Let us note that it is only because of vortices one has negative contribution to the electric part of the energy.

Now for the magnetic part. Since b^2 is the singlet part of B^2, by definition in every gauge invariant state $\langle b^2 \rangle = \langle B^2 \rangle$. We will therefore calculate $\langle B^2 \rangle$. Here one should be a little careful. Since B^2 itself is not gauge invariant, one cannot use Eq. (3.4), but rather explicitly keep both integrals, over ϕ and η:

$$\begin{aligned} \langle b^2 \rangle &= Z^{-1} \int D\phi' D\phi'' D A_i B^2 \exp\{-\tfrac{1}{2}[A_i^{\phi'} G^{-1} A_i^{\phi'} + A_i^{\phi''} G^{-1} A_i^{\phi''}]\} \\ &= Z^{-1} \int D\phi\, D\eta\, D A_i \left[\epsilon_{ij} \partial_i \left(A_j - \frac{1}{2g}(\phi - \eta) \right) \right]^2 \exp\{-\tfrac{1}{2}[A_i^{\phi} G^{-1} A_i^{\phi} + A_i G^{-1} A_i]\} , \end{aligned} \tag{3.20}$$

where the factor Z contains an extra factor of the volume of the gauge group relative to Eq. (3.5), and as previously $\phi = \phi' - \phi''$ and $\eta = \phi' + \phi''$. The linear term in η vanishes due to the symmetry of the measure under transformation $\eta \to -\eta$. The term quadratic in η is independent of the variational parameter G. It does not contribute to the minimization equations, and we omit it in the following. We therefore obtain

$$\begin{aligned} \langle b^2 \rangle &= Z^{-1} \int D\phi\, D\eta\, D A_i \left[\epsilon_{ij} \partial_i \left(A_j - \frac{1}{2g} \phi \right) \right]^2 \exp\{-\tfrac{1}{2}[A_i^{\phi} G^{-1} A_i^{\phi} + A_i G^{-1} A_i]\} \\ &= Z_a^{-1} \int D A_i [\epsilon_{ij} \partial_i A_j]^2 \exp\{-A_i G^{-1} A_i\} = \frac{1}{2} \int d^2k\, k^2 G(k) . \end{aligned} \tag{3.21}$$

This is the same result as in the noncompact theory. In fact, this is precisely what one expects, since a compact state Ψ differs from a noncompact one, $\tilde{\Psi}$ only by the presence of vortices, but b^2 by definition should not feel their presence.

Summarizing, we have the following expression for the expectation value of the energy density:

$$\frac{1}{V} \langle H \rangle = \frac{1}{4} \int \frac{d^2k}{(2\pi)^2} \left[G^{-1}(k) + k^2 G(k) - \frac{4\pi^2}{g^2} z k^{-2} G^{-2}(k) \right] . \tag{3.22}$$

The minimization equation is

$$\frac{1}{4}[k^2 - G^{-2}(k)] + \frac{\pi^2}{g^2} \left[2zk^{-2} G^{-3}(k) - 4\pi^2 \frac{\delta z}{\delta G(k)} \int \frac{d^2p}{(2\pi)^2} p^{-2} G^{-2}(p) \right] = 0 . \tag{3.23}$$

From Eq. (3.10) one finds

$$\frac{\delta z}{\delta G(k)} = \frac{1}{4g^2}k^{-2}G^{-2}(k)z \; . \tag{3.24}$$

Assuming perturbative behavior of G at large momenta $[G(k) \to k^{-1}]$, the ratio of the fourth term in Eq. (3.23) to the third term is of order

$$\frac{\delta z}{\delta G(k)} \frac{\int d^2pp^{-2}G^{-2}(p)}{2zk^{-2}G^{-3}(k)} \propto \frac{\Lambda^2}{g^2k} \; . \tag{3.25}$$

At weak coupling this is much greater than one for any value of momentum. We can therefore omit the third term from Eq. (3.23) and get a very simple equation for $G^{-2}(k)$,

$$k^2 - G^{-2}(k) = \frac{4\pi^4}{g^4}zk^{-2}G^{-2}(k)\int \frac{d^2p}{(2\pi)^2}p^{-2}G^{-2}(p) \; , \tag{3.26}$$

with the solution

$$G^{-2}(k) = \frac{k^4}{k^2+m^2} \; , \tag{3.27}$$

where

$$m^2 = \frac{4\pi^4}{g^4}z\int \frac{d^2k}{(2\pi)^2}k^{-2}G^{-2}(k) \; . \tag{3.28}$$

Using Eqs. (3.10) and (3.27) we get the equation for mass

$$m^2 = \frac{4\pi^4}{g^4}\Lambda^2 \exp\left(-\frac{\pi^2}{g^2}\int \frac{d^2p}{(2\pi)^2}\frac{1}{\sqrt{p^2+m^2}}\right) \times \int \frac{d^2k}{(2\pi)^2}\frac{k^2}{k^2+m^2} \; , \tag{3.29}$$

which in case of weak coupling $\Lambda/g^2 \gg 1$ can be simplified as

$$m^2 = \pi^3\frac{\Lambda^4}{g^4}\exp\left(-\frac{\pi}{2}\frac{\Lambda}{g^2}\right) \tag{3.30}$$

and because $m \ll g^2 \ll \Lambda$ one indeed could neglect the m dependence on the right-hand side of Eq. (3.29). It is clear that m is precisely the mass gap of the theory. Calculating, for example, the propagator of magnetic field, we find

$$\int d^2x\, e^{ikx}\langle b(x)b(0)\rangle = \tfrac{1}{2}(k^2+m^2)^{1/2} \; . \tag{3.31}$$

Note, that the dynamically generated mass we obtain in our approximation agrees with Polyakov's result [2].

Perhaps the most interesting question is whether our best variational VWF is confining. To answer this question we calculate the expectation value of the Wilson loop:

$$W_C = \left\langle \exp\left(ilg\oint_C A_i dx_i\right)\right\rangle = \left\langle \exp\left(ilg\int_S BdS\right)\right\rangle \; , \tag{3.32}$$

where l is an arbitrary integer and the integral is over the area S bounded by the loop C. We have written B rather than b, since this exponential operator is invariant under transformations $B(x) \to B(x) + 2\pi/g$, generated by the vortex operator:

$$W_C = Z_a^{-1}\int DA_i \exp\left\{-A_iG^{-1}A_i + ilg\int_S \epsilon_{ij}\partial_i A_j d^2x\right\} Z_\phi^{-1}Z_v^{-1}\int D\phi \exp\left\{-\frac{1}{4g^2}\partial_i\phi G^{-1}\partial_i\phi + i\frac{l}{2}\oint_C \partial_i\phi dx_i\right\} . \tag{3.33}$$

The first factor in weak coupling is simply

$$W_0 = \exp\left\{-\frac{l^2}{2}g^2\int_{x,y}\langle B(x)B(y)\rangle d^2x\, d^2y\right\} \; , \tag{3.34}$$

where the integral over both x and y is over the area S. In the limit of large S the leading piece in the exponential is

$$\frac{l^2}{2}g^2S\lim_{k\to 0}\frac{k^2}{2}G(k) = \frac{l^2}{4}g^2mS \; . \tag{3.35}$$

This term therefore leads to the area law behavior and gives the string tension

$$\sigma = \frac{l^2}{4}g^2m \tag{3.36}$$

and we see that area law $W_0 \sim \exp(-\sigma S)$ is a direct consequence of the nonzero mass gap m.

The second factor in Eq. (3.33) is different from unity only for odd l, since $\oint \partial_i\phi dl_i = 2\pi(n_+ - n_-)$, where n_+ (n_-) is the number of vortices (antivortices) inside the loop. For odd l it can be easily calculated:

$$W_v = \left\langle \exp\left(i\pi\int_S \rho(x)d^2x\right)\right\rangle = \int D\chi \exp\left(-2g^2\chi D^{-1}\chi + \int_x 2\Lambda^2\cos[\chi(x)-\alpha(x)]\right) \; , \tag{3.37}$$

where $\alpha(x)$ is the function, which vanishes for x outside the loop, and is equal to π for x inside the loop. At small coupling this can be calculated in the steepest descent approximation. The solution to the classical equations which contributes to the leading-order result is $\chi(x) = 0$. For this solution,

$$W_v = \exp\{-4zS\} \ , \tag{3.38}$$

where we again used the normal ordering prescription as in (3.15). Clearly this is a subleading correction to string tension (3.36), since $z \propto m^2$ and

$$z/g^2m \sim \sqrt{z/\Lambda^2} \sim \exp(-\pi\Lambda/4g^2) \ll 1 \ .$$

With this exponential accuracy, the string tension is therefore given in our approximation by Eq. (3.36).

IV. DISCUSSION

We have presented a simple variational calculation in the compact QED$_3$. Our variational ansatz was the direct adaptation of the ansatz of [1] to this theory. The trial wave functionals are explicitly gauge invariant under the compact gauge group. The integration over the gauge group is directly responsible for nontrivial dependence of the energy expectation value on the variational parameters, which leads to the generation of scale in the best variational state. The correlators and the Wilson loop calculated in the best variational state agree with known results.

It is illuminating at this point to interpret our calculation from the point of view of the three-dimensional Euclidean path integral. The vacuum wave functional of the theory can be represented in path-integral formalism. To get the vacuum WF $\Psi[A]$ one should calculate the path integral over the fields $A(x,t)$, with t varying from $-\infty$ to 0, with the boundary condition $A(x,t=0) = A(x)$. To be more precise, in calculating VEV of some operator $O(t=0)$, one should split the time coordinate of the plane with the time coordinate of the operator, so that one considers $\Psi[A(x,t=-\epsilon)]$ and $\Psi^*[A(x,t=\epsilon)]$ in the limit $\epsilon \to 0$.

The basic objects that appear in the Euclidean path integrals are monopoles, which in 3D are not propagating particles, but rather instantons. When described in terms of the vector potential, or noncompact field strength, a monopole has a Dirac string attached to it. It is clear that the vortices (antivortices) of the gauge function $\phi(x)$, in Eq. (2.3) correspond precisely to the intersections of the Dirac strings of the three-dimensional (3D) monopoles (antimonopoles) with the equal time plane at $t=0$. The positions of the Dirac strings are not physical in the compact theory, and only the position of the monopole itself is gauge invariant. In fact, for all monopoles that do not sit in the infinitesimally thin time slab between the planes $t=-\epsilon$ and ϵ, one can always choose the direction of the Dirac string such that it does not intersect the two planes. This precisely corresponds to expression equation (3.4). The combination that enters this path integral nontrivially is $\phi = \phi' - \phi''$. At the points, where both functions $\phi'(x)$ (which corresponds to Ψ) and $\phi''(x)$ (which corresponds to Ψ^*) have a vortex, $\phi(x)$ is regular. This is the situation, when a Dirac string intersects both planes $t = \pm\epsilon$. When a 3D monopole sits in the slab, only one of the functions ϕ' or ϕ'' has nonzero vorticity, and so does ϕ. The integration over $\phi(x)$ in Eq. (3.4) can be interpreted therefore as the direct contribution to the expectation value due to the monopoles at precisely the time $t=0$.

The fact that in this way one sees directly only the monopoles at $t=0$, does not mean of course that other monopoles are not taken into account in this approximation. Indeed, the "bare" interaction potential between the $t=0$ monopoles is $D(x)$ of Eq. (3.9). In the best variational state it is already short range, as follows from the solution for $G(k)$ [see Eq. (3.27)]. This is in accordance with the 3D picture, that the 3D monopole gas produces screening. Obviously, if one only looks at the thin slab, every monopole there will have an antimonopole partner, which sits nearby (inside the screening length) in the third direction. The 2D monopole gas will therefore be screened by the 3D interaction, even before the interaction of the 2D monopoles between themselves is taken into account. This is perfectly consistent with our calculation. It is interesting to note, that even though this 2D interaction produces additional screening [the cosine term in the effective theory (3.13)], it is the 3D screening that is responsible for the area law of the Wilson loop, as is clear from the calculation of the string tension in (3.34). In fact, if one takes for $G(k)$ the noncompact expression, both the leading part W_0, Eq. (3.34), and the subleading part W_v, (3.37), have the perimeter law behavior. The string tension in W_0 vanishes, because in this case $\lim_{k\to 0} k^2G(k) = 0$. In W_v in this case $D^{-1}(k^2=0) = 0$, and for large loops the classical equations of motion, which follow from Eq. (3.37), apart from $\chi = 0$ have another solution, which leads to the perimeter dependence of W_v. The existence of the cosine term in the interaction (which appears due to the 2D screening) does not preclude the existence of this extra solution.

Finally, we note that the calculation presented here can be extended to compact QED in 3+1 dimensions. In this case the vortex gas part of the partition function will be replaced by the gas of vortex loops with the interaction between the loop elements dependent on the variational function G. It is well known that compact QED$_4$ has a phase transition at a finite coupling constant. There is a good chance that this phase transition will be seen in the present approximation. Consider variational functions G, for which ∂^2G^2 is short range, as in Eq. (3.27). The interaction between the strings is then short range. In this case the standard energy vs entropy argument is telling us that at large g^2 the strings are condensed. The string gas contribution to the vacuum energy will then be sizable. Since this contribution has a negative sign, this situation will be energetically favored. Therefore at large g^2 one expects the best variational state to have a short ranged ∂^2G^2, and therefore the mass scale will be generated dynamically. At small g^2 the vortex rings do not condense even for short-range interaction between them. Their contribution to the vacuum energy will be negligible, and one expects that the best variational vacuum will be determined by the contributions from the A_i and $\tilde{\phi}$ integrals, which will lead to the same solution as in the noncompact theory.

Note added in proof. After this paper was submitted for publication we learned about Ref. [6], where a similar variational calculation was applied to the lattice formulation of the compact QED_3. Our results agree with those of [6]. We thank Ben Svetitsky for bringing this reference to our attention and for interesting correspondence.

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